

3D STEADY GRADIENT RICCI SOLITONS WITH LINEAR CURVATURE DECAY

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ABSTRACT. In this note, we prove that a 3-dimensional steady Ricci soliton is rotationally symmetric if its scalar curvature $R(x)$ satisfies

$$\frac{C_0^{-1}}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)}$$

for some constant $C_0 > 0$, where $\rho(x)$ denotes the distance from a fixed point x_0 . Our result doesn't assume that the soliton is κ -noncollapsed.

1. INTRODUCTION

In his celebrated paper [13], Perelman conjectured that *all 3-dimensional κ -noncollapsed steady (gradient) Ricci solitons must be rotationally symmetric*. The conjecture is solved by Brendle in 2012 [1]. For a general dimension $n \geq 3$, under an extra condition that the soliton is asymptotically cylindrical, Brendle also proves that any κ -noncollapsed steady Ricci soliton with positive sectional curvature must be rotationally symmetric [2]. In general, it is still open *whether an n -dimensional κ -noncollapsed steady Ricci soliton with positive curvature operator is rotationally symmetric for $n \geq 4$* . For κ -noncollapsed steady Kähler-Ricci solitons with nonnegative bisectional curvature, the authors have recently proved that they must be flat [10], [11].

Recall from [2],

Definition 1.1. *An n -dimensional steady Ricci soliton (M, g, f) is called asymptotically cylindrical if the following holds:*

(i) *Scalar curvature $R(x)$ of g satisfies*

$$\frac{C_0^{-1}}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)}, \quad \forall \rho(x) \geq r_0,$$

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where $C_0 > 0$ is a constant and $\rho(x)$ denotes the distance of x from a fixed point x_0 .

(ii) Let p_m be an arbitrary sequence of marked points going to infinity. Consider rescaled metrics $g_m(t) = r_m^{-1} \phi_{r_m t}^* g$, where $r_m R(p_m) = \frac{n-1}{2} + o(1)$ and ϕ_t is a one-parameter subgroup generated by $X = -\nabla f$. As $m \rightarrow \infty$, flows $(M, g_m(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(\mathbb{R} \times \mathbb{S}^{n-1}(1), \tilde{g}(t)), t \in (0, 1)$. The metric $\tilde{g}(t)$ is given by

$$(1.1) \quad \tilde{g}(t) = dr^2 + (n-2)(2-2t)g_{\mathbb{S}^{n-1}(1)},$$

where $\mathbb{S}^{n-1}(1)$ is the unit sphere of euclidean space.

In this note, we discuss 3-dimensional steady (gradient) Ricci solitons without assuming the κ -noncollapsed condition.¹ We prove

Theorem 1.2. *Let (M, g, f) be a 3-dimensional steady Ricci soliton. Then, it is rotationally symmetric if the scalar curvature $R(x)$ of (M, g, f) satisfies*

$$(1.2) \quad \frac{C_0^{-1}}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)},$$

for some constant $C > 0$, where $\rho(x)$ denotes the distance from a fixed point x_0 .

Under the condition (1.2), we need to check the property (ii) in Definition 1.1 to prove Theorem 1.2. Actually, we show that for any sequence $p_i \rightarrow \infty$, there exists a subsequence $p_{i_k} \rightarrow \infty$ such that

$$(M, g_{p_{i_k}}(t), p_{i_k}) \rightarrow (\mathbb{R} \times \mathbb{S}^2, \tilde{g}(t), p_\infty), \text{ for } t \in (-\infty, 1),$$

where $g_{p_{i_k}}(t) = R(p_{i_k})g(R^{-1}(p_{i_k})t)$ and $(\mathbb{R} \times \mathbb{S}^2, \tilde{g}(t))$ is a shrinking cylinders flow, i.e.

$$\tilde{g}(t) = dr^2 + (2-2t)g_{\mathbb{S}^2}.$$

As in [10], we study the geometry of neighborhood $M_{p,k} = \{x \in M \mid f(p) - \frac{k}{\sqrt{R(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{R(p)}}\}$ around level set $\Sigma_r = \{f(x) = f(p) = r\}$ for any $p \in M$. We are able to give a uniform injective radius estimate for $(M, R_{p_i}g)$ at each sequence of p_i . Then we can still get a limit flow for rescaled flows $(M, g_{p_i}(t))$, which will split off a line. By using a classification result of Daskalopoulos-Hamilton-Sesum for ancient flows on a compact surface [8], we finish the proof of Theorem 1.2.

We remark that the curvature condition in Theorem 1.2 cannot be removed, since there does exist a 3-dimensional non-flat steady Ricci soliton with exponential curvature decay. For example, $(\mathbb{R}^2 \times \mathbb{S}^1, g_{cigar} + ds^2)$, where

¹It is proved by Chen that any 3-dimensional ancient solution has nonnegative sectional curvature [7].

$(\mathbb{R}^2, g_{cigar})$ is a cigar soliton. Also, Theorem 1.2 is not true for dimension $n \geq 4$ by Cao's examples of steady Kähler-Ricci solitons with positive sectional curvature [3].

At last, we remark that it is still open *wether there exists a 3-dimensional collapsed steady Ricci soliton with positive curvature*. Hamilton has conjectured that *there should exist a family of collapsed 3-dimensional complete gradient steady Ricci solitons with positive curvature and S^1 -symmetry* (cf. [5]). Our result shows that the curvature of Hamilton's examples could not have a linear decay.

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2. POSITIVITY OF RICCI CURVATURE

(M, g, f) is called a gradient steady Ricci soliton if a Riemannian metric g on M satisfies

$$(2.1) \quad \text{Ric}(g) = \nabla^2 f,$$

for some smooth function f . We first show the positivity of Ricci curvature of (M, g, f) under (1.2) of Theorem 1.2.

Lemma 2.1. *Under (1.2), (M, g, f) has positive sectional curvature.*

Proof. We need to show that (M, g) has positive Ricci curvature. On the contrary, (M, g) locally splits off a flat piece of line by Shi's splitting theorem [14]. Then, the universal cover $(\widetilde{M}, \widetilde{g})$ of (M, g) is isometric to a product Riemannian manifold of a real line and the cigar soliton. Namely, $(\widetilde{M}, \widetilde{g}) = (\mathbb{R}^2 \times \mathbb{R}, g_{cigar} + ds^2)$. Let $\pi : \widetilde{M} \rightarrow M$ be a universal covering. We fix $x_0 \in M$ and $\widetilde{x}_0 \in \widetilde{M}$ such that $\pi(\widetilde{x}_0) = x_0$. For any $x \in M$ and $\widetilde{x} \in \widetilde{M}$ such that $\pi(\widetilde{x}) = x$, one sees

$$(2.2) \quad \rho(x, x_0) \leq \widetilde{\rho}(\widetilde{x}, \widetilde{x}_0),$$

where ρ and $\widetilde{\rho}$ are the distance functions w.r.t g and \widetilde{g} respectively. Let $\{\widetilde{x}_i\}_{i \geq 1}$ be a sequence of points so that $\widetilde{x}_i = (p_i, 0) \in \mathbb{R}^2 \times \mathbb{R}$ tend to infinity. Then, one may check that

$$(2.3) \quad \widetilde{R}(\widetilde{x}_i) \rho(\widetilde{x}_i, \widetilde{x}_0) \rightarrow 0, \text{ as } i \rightarrow \infty.$$

Since $R(x_i) = \widetilde{R}(\widetilde{x}_i) \rightarrow 0$, where $x_i = \pi(\widetilde{x}_i)$, we see $d(x_i, x_0) \rightarrow \infty$ by (1.2). Again by (1.2) and (2.2), we get

$$(2.4) \quad C_1 \leq R(x_i) d(x_i, x_0) \leq \widetilde{R}(\widetilde{x}_i) d(\widetilde{x}_i, \widetilde{x}_0).$$

This is a contradiction to (2.3). Hence, the lemma is proved. \square

Corollary 2.2. *(M, g, f) in Theorem 1.2 has a unique equilibrium point o , i.e., $\nabla f(o) = 0$. As a consequence, $\Sigma_r = \{f(x) = r\}$ is diffeomorphic to \mathbb{S}^2 , for any $r > f(o)$.*

Proof. Note that

$$(2.5) \quad |\nabla f|^2 + R = A.$$

By taking covariant derivatives on both sides of (2.5), it follows

$$(2.6) \quad 2\text{Ric}(\nabla f, \nabla f) = -\langle \nabla R, \nabla f \rangle.$$

On the the hand, by (1.2), there exists a point o such that

$$\sup_M R(x) = R(o) = R_{\max}.$$

In particular, $\nabla R(o) = 0$. Thus

$$\text{Ric}(\nabla f, \nabla f) = 0.$$

By Lemma 2.1, $\nabla f(o) = 0$. The uniqueness also follows from the positivity of Ricci curvature.

By the Morse theorem, $\Sigma_r = \{f(x) = r > f(o)\}$ is diffeomorphic to \mathbb{S}^2 (cf. [10], Lemma 2.1). \square

3. GEOMETRY OF $M_{p,k}$

For any $p \in M$ and number $k > 0$, we set

$$M_{p,k} = \{x \in M \mid f(p) - \frac{k}{\sqrt{R(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{R(p)}}\}.$$

Let $g_p = R(p)g$ be a rescaled metric and denote $B(p, r; g_p)$ a r -geodesic ball centered at p with respect to g_p . Then by Corollary 2.2, we have (cf. [10], Lemma 3.3)

Lemma 3.1. *Under (1.2), for any $p \in M$ and number $k > 0$ with $f(p) - \frac{k}{\sqrt{R(p)}} > f(o)$, it holds*

$$(3.1) \quad B(p, \frac{k}{\sqrt{R_{\max}}}; g_p) \subset M_{p,k}.$$

By Lemma 3.1, we prove

Lemma 3.2. *Under (1.2), there exists a constant C such that*

$$(3.2) \quad \frac{|\Delta R|(p)}{R^2(p)} \leq C, \quad \forall p \in M.$$

Proof. Fix any $p \in M$ with $f(p) \geq r_0 \gg 1$. Then

$$(3.3) \quad |f(x) - f(p)| \leq \frac{1}{\sqrt{R(p)}}, \quad \forall x \in M_{p,1}.$$

It is known by [4],

$$(3.4) \quad c_1 \rho(x) \leq f(x) \leq c_2 \rho(x), \quad \forall \rho(x) \geq r_0.$$

Thus by (1.2), (3.3) and (3.4), we get

$$c_2 \rho(x) \geq f(p) - \frac{1}{\sqrt{R(p)}} \geq c_1 \rho(p) - \sqrt{C_0 \rho(p)}.$$

It follows

$$(3.5) \quad \frac{R(x)}{R(p)} \leq C_0^2 \frac{\rho(p)}{\rho(x)} \leq \frac{2c_2 C_0^2}{c_1}, \quad \forall x \in M_{p,1}.$$

On the other hand, by (3.1), we have

$$B(p, \frac{1}{\sqrt{R_{\max}}}; g_p) \subseteq M_{p,1}.$$

Hence

$$(3.6) \quad R(x) \leq C' R(p), \quad \forall x \in B(p, \frac{1}{\sqrt{R_{\max}}}; g_p).$$

Let ϕ_t be generated by $-\nabla f$. Then $g(t) = \phi_t^* g$ satisfies the Ricci flow,

$$(3.7) \quad \frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)).$$

Also rescaled flow $g_p(t) = R(p)g(R^{-1}(p)t)$ satisfies (3.7). Since the Ricci curvature is positive,

$$B(p, \frac{1}{\sqrt{R_{\max}}}; g_p(-t)) \subseteq B(p, \frac{1}{\sqrt{R_{\max}}}; g_p(0)), \quad t \in [-1, 0].$$

Combining with (3.6), we get

$$(3.8) \quad R_{g_p(t)}(x) \leq C', \quad \forall x \in B(p, \frac{1}{\sqrt{R_{\max}}}; g_p(0)), \quad t \in [-1, 0].$$

Thus, by Shi's higher order estimates, we obtain

$$|\Delta_{g_p(t)} R_{g_p(t)}|(x) \leq C'_1, \quad \forall x \in B(p, \frac{1}{2\sqrt{R_{\max}}}; g_p(-1)), \quad t \in [-\frac{1}{2}, 0].$$

It follows

$$|\Delta R|(x) \leq C'_1 R^2(p), \quad \forall x \in B(p, \frac{1}{2\sqrt{R_{\max}}}; g_p(-1)).$$

In particular, we have

$$|\Delta R|(p) \leq C'_1 R^2(p), \quad \text{as } \rho(p) \geq r_0.$$

The lemma is proved. \square

Remark 3.3. Under (1.2), by the same argument as in the proof of Lemma 3.2, for each $k \in \mathbb{N}$, there exists a constant $C(k)$ such that

$$(3.9) \quad \frac{|\nabla^k R|(p)}{R^{\frac{k+2}{2}}(p)} \leq C(k), \quad \forall p \in M.$$

Next, we want to show that $M_{p,k}$ is bounded by a finite ball $B(p, Ck; g_p)$, where C is a uniform constant. We need to use the Gauss formula,

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle,$$

where $X, Y, Z, W \in T\Sigma_r$ and $B(X, Y) = (\nabla_X Y)^\perp$. Note that

$$\begin{aligned} B(X, Y) &= \langle \nabla_X Y, \nabla f \rangle \cdot \frac{\nabla f}{|\nabla f|^2} \\ &= [\nabla_X \langle Y, \nabla f \rangle - \langle Y, \nabla_X \nabla f \rangle] \cdot \frac{\nabla f}{|\nabla f|^2} \\ &= -\text{Ric}(X, Y) \cdot \frac{\nabla f}{|\nabla f|^2}. \end{aligned}$$

We choose a normal basis $\{e_1, e_2\}$ on (Σ_r, \bar{g}) with the induced metric \bar{g} . Then $\{e_1, e_2, \frac{\nabla f}{|\nabla f|}\}$ spans a normal basis of (M, g) . Thus

$$\begin{aligned} R_{11} &= \overline{R}_{11} + R\left(\frac{\nabla f}{|\nabla f|}, e_1, e_1, \frac{\nabla f}{|\nabla f|}\right) - \frac{R_{11}R_{22} - R_{12}R_{21}}{|\nabla f|^2}, \\ R_{22} &= \overline{R}_{22} + R\left(\frac{\nabla f}{|\nabla f|}, e_2, e_2, \frac{\nabla f}{|\nabla f|}\right) - \frac{R_{11}R_{22} - R_{12}R_{21}}{|\nabla f|^2}. \end{aligned}$$

Since (Σ_r, \bar{g}) is a surface, $K = \overline{R}_{11} = \overline{R}_{22}$. Hence, we get

Lemma 3.4. The Gauss curvature of (Σ_r, \bar{g}) is given by

$$(3.10) \quad K = \frac{R}{2} - \frac{\text{Ric}(\nabla f, \nabla f)}{|\nabla f|^2} + \frac{R_{11}R_{22} - R_{12}R_{21}}{|\nabla f|^2}.$$

Lemma 3.5. Under (1.2), there exists a uniform $B > 0$ such that the following is true: for any $k \in \mathbb{N}$, there exists $\bar{r}_0 = \bar{r}_0(k)$ such that

$$(3.11) \quad M_{p,k} \subset B(p, 2\pi\sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; g_p), \quad \forall \rho(p) \geq \bar{r}_0.$$

Proof. By (1.2) and (3.4), we have

$$(3.12) \quad \frac{R_{\max}}{2} \leq |\nabla f|^2(x) \leq R_{\max}, \quad \forall x \in M_{p,k},$$

as long as $\rho(p) \geq r_0 > 1$. Then by Lemma 3.4 and Lemma 3.2, we get

$$\begin{aligned}
|K - \frac{R}{2}| &= \left| -\frac{\text{Ric}(\nabla f, \nabla f)}{|\nabla f|^2} + \frac{R_{11}R_{22} - R_{12}R_{21}}{|\nabla f|^2} \right| \\
&\leq \frac{|\langle \nabla R, \nabla f \rangle|}{2|\nabla f|^2} + \frac{R^2}{|\nabla f|^2} \\
&\leq \frac{|\Delta R + 2|\text{Ric}|^2|}{2|\nabla f|^2} + \frac{R^2}{|\nabla f|^2} \\
&\leq \frac{(C+4)R^2}{R_{\max}}.
\end{aligned}$$

It follows

$$(3.13) \quad \frac{R(x)}{4} \leq K(x) \leq \frac{3R(x)}{4}, \quad \forall x \in M_{p,k}, \quad \rho(p) \geq r_0.$$

On the other hand, by (1.2), (3.3) and (3.4), we see

$$\begin{aligned}
c_2^{-1} \left(c_1 \rho(p) - k \sqrt{\rho(p) C_0} \right) &\leq \rho(x) \\
(3.14) \quad &\leq c_1^{-1} (c_2 \rho(p) + k \sqrt{\rho(p) C_0}), \quad \forall x \in M_{p,k},
\end{aligned}$$

as long as $\rho(p) \geq r_0$. Then similar to (3.5), there exists a $\bar{r}_0 \geq r_0$ such that

$$(3.15) \quad R(x) \geq \frac{c_1}{2c_2 C_0^2} R(p), \quad \forall x \in M_{p,k}.$$

Thus by (3.13), we get

$$\bar{R}_{ij} \geq B^{-1} R(p) \bar{g}_{ij}, \quad \forall x \in \Sigma_{f(p)}, \quad \rho(p) \geq \bar{r}_0,$$

where $B > 0$ is a uniform constant. By the Myer's theorem, the diameter of $\Sigma_{f(p)}$ is bounded by

$$\text{diam}(\Sigma_{f(p)}, g) \leq \text{diam}(\Sigma_{f(p)}, \bar{g}_{f(p)}) \leq 2\pi \sqrt{\frac{B}{R(p)}}.$$

As a consequence,

$$(3.16) \quad \Sigma_{f(p)} \subset B(p, 2\pi\sqrt{B}; R(p)g).$$

For any $q \in M_{p,k}$, there exists $q' \in \Sigma_{f(p)}$ such that $\phi_s(q) = q'$ for some $s \in \mathbb{R}$. Then by (3.16) and (3.12), we have

$$\begin{aligned}
d(q, p) &\leq d(q', p) + d(q, q') \\
&\leq \text{diam}(\Sigma_{f(p)}, g) + \mathcal{L}(\phi_\tau|_{[0, s]}) \\
&\leq 2\pi\sqrt{\frac{B}{R(p)}} + \left| \int_0^s \left| \frac{d\phi_\tau(q)}{d\tau} \right| d\tau \right| \\
&= 2\pi\sqrt{\frac{B}{R(p)}} + \int_0^s |\nabla f(\phi_\tau(q))| d\tau \\
&\leq 2\pi\sqrt{\frac{B}{R(p)}} + \int_0^s |\nabla f(\phi_\tau(q))|^2 \cdot \frac{2}{\sqrt{R_{\max}}} d\tau \\
&= 2\pi\sqrt{\frac{B}{R(p)}} + \left| \int_0^s \frac{d(f(\phi_\tau(q)))}{d\tau} \cdot \frac{2}{\sqrt{R_{\max}}} d\tau \right| \\
&\leq 2\pi\sqrt{\frac{B}{R(p)}} + |f(q) - f(p)| \cdot \frac{2}{\sqrt{R_{\max}}} \\
&\leq \left(2\pi\sqrt{B} + \frac{2k}{\sqrt{R_{\max}}} \right) \cdot \frac{1}{\sqrt{R(p)}}.
\end{aligned}$$

Thus

$$M_{p,k} \subset B(p, 2\pi\sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; R(p)g).$$

The lemma is proved. \square

By Lemma 3.5, we get the following volume estimate of $B(p, s; g_p)$.

Proposition 3.6. *Under (1.2) of Theorem 1.2, there exists s_0 and $c > 0$ such that*

$$(3.17) \quad \text{Vol}B(p, s; g_p) \geq cs^3, \quad \forall s \leq s_0 \text{ and } \rho(p) \geq r_0 \gg 1.$$

Moreover, the injective radius of (M, g_p) at p has a uniform lower bound $\delta > 0$, i.e.,

$$(3.18) \quad \text{inj}(p, g_p) \geq \delta, \quad \forall \rho(p) \geq r_0.$$

Proof. By Lemma 3.5, we have

$$M_{p,1} \subset B(p, 2\pi\sqrt{B} + \frac{2}{\sqrt{R_{\max}}}; g_p).$$

In the following, we give an estimate of $\text{Vol}(\Sigma_l, \bar{g})$ for any l with $f(p) - \frac{1}{\sqrt{R(p)}} \leq l \leq f(p) + \frac{1}{\sqrt{R(p)}}$.

By (3.5) and (3.15), we see

$$C_1^{-1} \leq \frac{R(x)}{R(p)} \leq C_1, \quad \forall \rho(p) \geq r_0 \text{ and } x \in M_{p,1}.$$

By (3.13), it follows that the Gauss curvature K_l of $(\Sigma_l, g_p|_{\Sigma_l})$ satisfies

$$\frac{1}{4C_1} \leq K_l \leq \frac{3C_1}{4}.$$

Thus

$$\text{Vol}(\Sigma_l, \bar{g}) = \frac{1}{R(p)} \text{Vol}(\Sigma_l, g_p|_{\Sigma_l}) \geq \frac{64\pi C_1}{R(p)}.$$

By the Co-Area formula, we get

$$\begin{aligned} \text{Vol}(M_{p,1}, g) &= \int_{f(p) - \frac{1}{\sqrt{R(p)}}}^{f(p) + \frac{1}{\sqrt{R(p)}}} \frac{\text{Vol}(\Sigma_l, \bar{g})}{|\nabla f|} dl \\ &\geq 128\pi C_1 R_{\max}^{-\frac{1}{2}} R^{-\frac{3}{2}}(p). \end{aligned}$$

Hence

$$(3.19) \quad \text{Vol}(B(p, 2\pi\sqrt{B} + \frac{2}{\sqrt{R_{\max}}}; g_p)) \geq \text{Vol}(M_{p,1}, g_p) \geq 128\pi C_1 R_{\max}^{-\frac{1}{2}}.$$

By the volume comparison theorem, we derive from (3.19),

$$\begin{aligned} \frac{\text{Vol}(B(p, s; g_p))}{s^3} &\geq \frac{\text{Vol}(B(p, 2\pi\sqrt{B} + \frac{2}{\sqrt{R_{\max}}}; g_p))}{(2\pi\sqrt{B} + \frac{2}{\sqrt{R_{\max}}})^3} \\ &\geq \frac{128\pi C_1 R_{\max}^{-\frac{1}{2}}}{(2\pi\sqrt{B} + \frac{2}{\sqrt{R_{\max}}})^3}, \end{aligned}$$

for any $s \leq 2\pi\sqrt{B} + \frac{2}{\sqrt{R_{\max}}}$. This proves (3.17). By (3.17), we can apply a result of Cheeger-Gromov-Taylor for Riemannian manifolds with bounded curvature to get the injective radius estimate (3.18) immediately [6]. \square

4. PROOF OF THEOREM 1.2

First we prove the following convergence of rescaled flows.

Lemma 4.1. *Under (1.2), let $p_i \rightarrow \infty$. Then by taking a subsequence of p_i if necessary, we have*

$$(M, g_{p_i}(t), p_i) \rightarrow (\mathbb{R} \times N, \tilde{g}(t); p_\infty), \text{ for } t \in (-\infty, 0],$$

where $g_{p_i}(t) = R(p_i)g(R^{-1}(p_i)t)$, $\tilde{g}(t) = ds \otimes ds + g_N(t)$ and $(N, g_N(t))$ is an ancient solution of Ricci flow on N .

Proof. For a fixed \bar{r} , as in (3.5), it is easy to see that there exists a uniform C_1 independent of \bar{r} such that

$$(4.1) \quad R(x) \leq C_1 R(p_i), \quad \forall x \in M_{p_i, \bar{r} \sqrt{R_{\max}}}$$

as long as i is large enough. By Lemma 3.1, it follows

$$R_{g_{p_i}}(x) \leq C_1, \quad \forall x \in B(p_i, \bar{r}; g_{p_i}),$$

where $g_{p_i} = g_{p_i}(0)$. Since the scalar curvature is increasing along the flow, we get

$$\begin{aligned} |\text{Rm}_{g_{p_i}(t)}(x)|_{g_{p_i}(t)} &\leq 3R_{g_{p_i}(t)}(x) \\ &\leq 3R_{g_{p_i}}(x) \leq 3C_1, \quad \forall x \in B(p_i, \bar{r}; g_{p_i}), \quad t \in (-\infty, 0]. \end{aligned}$$

Thus together with the injective radius estimate in Proposition 3.6, we can apply the Hamilton compactness theorem to see that $g_{p_i}(t)$ converges subsequently to a limit flow $(\widetilde{M}, \tilde{g}(t); p_\infty)$ on $t \in (-\infty, 0]$ [12]. Moreover, the limit flow has uniformly bounded curvature. It remains to prove the splitting property.

By Remark 3.3, we have

$$|\text{Ric}|(x) \leq CR(x), \quad \forall x \in B(p_i, \bar{r}; g_{p_i}).$$

It follows from (4.1),

$$|\text{Ric}|(x) \leq CR(p_i), \quad \forall x \in B(p_i, \bar{r}; g_{p_i}).$$

Let $X_{(i)} = R(p_i)^{-\frac{1}{2}} \nabla f$. Then

$$\begin{aligned} \sup_{B(p_i, r_0; g_{p_i})} |\nabla_{(g_{p_i})} X_{(i)}|_{g_{p_i}} &= \sup_{B(p_i, r_0; g_{p_i})} \frac{|\text{Ric}|}{\sqrt{R(p_i)}} \\ &\leq C \sqrt{R(p_i)} \rightarrow 0. \end{aligned}$$

On the other hand, by Remark 3.3, we also have

$$\sup_{B(p_i, r_0; g_{p_i})} |\nabla_{(g_{p_i})}^m X_{(i)}|_{g_{p_i}} \leq C(n) \sup_{B(p_i, r_0; g_{p_i})} |\nabla_{(g_{p_i})}^{m-1} \text{Ric}(g_{p_i})|_{g_{p_i}} \leq C_1.$$

Thus $X_{(i)}$ converges subsequently to a parallel vector field $X_{(\infty)}$ on $(\widetilde{M}, \tilde{g}(0))$. Moreover,

$$|X_{(i)}|_{g_{p_i}}(x) = |\nabla f|(p_i) = \sqrt{R_{\max}} + o(1) > 0, \quad \forall x \in B(p_i, r_0; g_i),$$

as long as $f(p_i)$ is large enough. This implies that $X_{(\infty)}$ is non-trivial. Hence, $(\widetilde{M}, \tilde{g}(t))$ locally splits off a piece of line along $X_{(\infty)}$. It remains to show that $X_{(\infty)}$ generates a line through p_∞ .

By Lemma 3.5,

$$M_{p_i, k} \subset B(p_i, 2\pi\sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; g_{p_i}(0)), \quad \forall p_i \rightarrow \infty.$$

Let $\gamma_{i,k}(s)$, $s \in (-D_{i,k}, E_{i,k})$ be an integral curve generated by $X_{(i)}$ through p_i , which restricted in $M_{p,k}$. Then $\gamma_{i,k}(s)$ converges to a geodesic $\gamma_\infty(s)$ generated by $X_{(\infty)}$ through p_∞ , which restricted in $B(p_\infty, 2\pi\sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; \tilde{g}(0))$. If let $L_{i,k}$ be lengths of $\gamma_{i,k}(s)$ and $L_{\infty,k}$ length of $\gamma_\infty(s)$,

$$\begin{aligned} L_{i,k} &= \int_{-D_{i,k}}^{E_{i,k}} |\nabla f|_{g_{p_i}(0)} ds = \int_{f(p_i) - \frac{k}{\sqrt{R(p_i)}}}^{f(p_i) + \frac{k}{\sqrt{R(p_i)}}} \sqrt{R(p_i)} \|\nabla f\|_g ds \\ &\geq R_{\max}^{-\frac{1}{2}} \int_{f(p_i) - \frac{k}{\sqrt{R(p_i)}}}^{f(p_i) + \frac{k}{\sqrt{R(p_i)}}} \sqrt{R(p_i)} \|\nabla f\|_g^2 ds \\ &= R_{\max}^{-\frac{1}{2}} \int_{f(p_i) - \frac{k}{\sqrt{R(p_i)}}}^{f(p_i) + \frac{k}{\sqrt{R(p_i)}}} \sqrt{R(p_i)} df = 2R_{\max}^{-\frac{1}{2}} k, \end{aligned}$$

and so,

$$L_{\infty,k} \geq \frac{1}{2} L_{i,k} \geq R_{\max}^{-\frac{1}{2}} k.$$

Thus $X_{(\infty)}$ generates a line $\gamma_\infty(s)$ through p_∞ as $k \rightarrow \infty$. As a consequence, $(\tilde{M}, \tilde{g}(0))$ splits off a line and so does the flow $(\tilde{M}, \tilde{g}(t); p_\infty)$. The lemma is proved. \square

Next we estimate the curvature of $(N, g_N(t))$.

Lemma 4.2. *Under (1.2), there exists a constant C independent of t such that the scalar curvature $R_N(t)$ of $(N, g_N(t))$ satisfies*

$$(4.2) \quad \frac{R_N(x, t)}{R_N(y, t)} \leq C, \quad \forall x, y \in N, \quad t \in (-\infty, 0].$$

Proof. Let $\tilde{R}(x, t)$ be the scalar curvature of $(\mathbb{R} \times N, \tilde{g}(t))$. It suffices to prove the following is true:

$$(4.3) \quad \frac{\tilde{R}(x, t)}{\tilde{R}(y, t)} \leq C, \quad \forall x, y \in \mathbb{R} \times N, \quad t \in (-\infty, 0],$$

for some constant C . For any $x, y \in \mathbb{R} \times N$, we choose $\bar{r} > 0$ such that $x, y \in B(p_\infty, \bar{r}; \tilde{g}(0))$. By the convergence of $g_{p_i}(t)$, there are sequences $\{x_i\}$ and $\{y_i\}$ in $B(p_i, \bar{r}; g_{p_i}(0))$ such that x_i and y_i converge to x and y in the Cheeger-Gromov sense, respectively. By Lemma 3.1, we have

$$x_i, y_i \subseteq B(p_i, \bar{r}; g_{p_i}(0)) \subseteq M_{p_i, \bar{r}\sqrt{R_{\max}}}.$$

Thus

$$(4.4) \quad f(x_i) = (1 + o(1))f(p_i) \text{ and } f(y_i) = (1 + o(1))f(p_i), \text{ as } p_i \rightarrow \infty.$$

On the other hand, for a fixed $t < 0$,

$$\frac{f(\phi_{R^{-1}(p_i)t}(x_i)) - f(x_i)}{|R^{-1}(p_i)t|} = \frac{\int_{R^{-1}(p_i)t}^0 |\nabla f|^2 ds}{|R^{-1}(p_i)t|} \rightarrow R_{\max}, \text{ as } p_i \rightarrow \infty$$

and

$$\frac{f(\phi_{R^{-1}(p_i)t}(y_i)) - f(y_i)}{|R^{-1}(p_i)t|} = \frac{\int_{R^{-1}(p_i)t}^0 |\nabla f|^2 ds}{|R^{-1}(p_i)t|} \rightarrow R_{\max}, \text{ as } p_i \rightarrow \infty.$$

By (4.4) and the fact

$$C_1 \leq R(x)f(x) \leq C_2, \forall f(x) \gg 1,$$

we get

$$\frac{f(\phi_{R^{-1}(p_i)t}(x_i))}{f(\phi_{R^{-1}(p_i)t}(y_i))} \rightarrow 1, \text{ as } p_i \rightarrow \infty.$$

It follows

$$\frac{R(\phi_{R^{-1}(p_i)t}(x_i))}{R(\phi_{R^{-1}(p_i)t}(y_i))} \leq \frac{C_2}{C_1}.$$

Hence we obtain

$$\begin{aligned} \frac{R_N(x, t)}{R_N(y, t)} &= \lim_{i \rightarrow \infty} \frac{R^{-1}(p_i)R(x_i, R^{-1}(p_i)t)}{R^{-1}(p_i)R(y_i, R^{-1}(p_i)t)} \\ &= \lim_{i \rightarrow \infty} \frac{R^{-1}(p_i)R(\phi_{R^{-1}(p_i)t}(x_i))}{R^{-1}(p_i)R(\phi_{R^{-1}(p_i)t}(y_i))} \leq \frac{C_2}{C_1}. \end{aligned}$$

This proves (4.3). □

The proof of Theorem 1.2 is completed by the following lemma.

Lemma 4.3. *$(N, g_N(t))$ in Lemma 4.1 is a shrinking spheres flow. Namely,*

$$(N, g_N(t)) = (\mathbb{S}^2, (2 - 2t)g_{\mathbb{S}^2}).$$

Proof. By Lemma 4.2, the Gauss curvature of $(N, g_N(0))$ has a uniform positive lower bound. Then N is compact by Myer's Theorem. On the other hand, by a classification theorem of Daskalopoulos-Hamilton-Sesum [8], an ancient solution on a compact surface N is either a shrinking spheres flow or a Rosenau solution. The Rosenau solution is obtained by compactifying $(\mathbb{R} \times \mathbb{S}^1(2), h(x, \theta, t) = u(x, t)(dx^2 + d\theta^2))$ by adding two points, where $u(x, t) = \frac{\sinh(-t)}{\cosh(x) + \cosh(t)}$ and $t \in (-\infty, 0)$. By a direct computation,

$$(4.5) \quad R_{h(t)} = \frac{\cosh(t) \cosh(x) + 1}{\sinh(-t)(\cosh(x) + \cosh(t))}.$$

It is easy to check that $R_{h(t)}$ doesn't satisfy (4.2) in Lemma 4.2 as $t \rightarrow -\infty$. Hence, $(N, g_N(t))$ must be a shrinking spheres flow on \mathbb{S}^2 . Note that $\tilde{R}(p_\infty, 0) = 1$. Then it is easy to see that $g_{\mathbb{S}^2}(t) = (2 - 2t)g_{\mathbb{S}^2}$. \square

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